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连通图的无符号拉普拉斯谱半径的界

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摘 要: 假设 $G=(V,E)$ 是 n 阶连通图,$\lambda_1(G)$ 表示 G 的无符号拉普拉斯谱半径。对于任意点 $v_i\in V(G)$,其在 G 中的邻点的集合用 N_i 表示,并且点 v_i 在 G 中的邻点的数目用 d_i 表示,G 的最大度用 $\Delta(G)$ 表示。文章展示了当连通图 G 的最大度为 $\Delta(G)<n-1$ 时,则 $\lambda_1(G)\geqslant\max\left\{m_i'+\left(1+\frac{(m_i'-1)^2}{d_{2,i}}\right)\frac{d_i}{m_i'}:v_i\in V(G)\right\}$,其中 $m_i'=\frac{\sum_{v_iv_j\in E(G)}\left(d_j-\left|N_i\cap N_j\right|\right)}{d_i}$;令 $d_{2,i}$ 表示 G 中与点 v_i 距离为2的点的数目,则 $\lambda_1(G)\geqslant\max\left\{\left(p_{ij}+\frac{\left(1-p_{ij}\right)^2}{p_{ij}\max\{1,d_i-1\}}\right)\left|N_i\cup N_j\right|:v_iv_j\in E(G),d_i\geqslant d_j\right\}$,其中 $p_{ij}=\frac{e\left(N_i,N_j-N_i\right)}{d_i\left(\left|N_i\cup N_j\right|-d_i\right)}$;如果 G 是简单连通图,则 $\lambda_1(G)\leqslant\sqrt{2\Delta(G)(\Delta(G)-1)+1}+1$.

关键词: 无符号拉普拉斯矩阵;谱半径;连通图
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令 G 是点集为 $V=\{v_1,v_2,\cdots,v_n\}$,边集为 $E(G)$ 的非空简单连通图。对于任意的点 $v_i\in V(G)$,设该点 v_i 的度为 d_i;该点 v_i 的邻点的集合为 N_i 以及与该点 v_i 的距离为2的点集合中的数目为 $d_{2,i}$. 特别地,用 $\Delta(G)$ 表示 G 的最大度,用 $\delta(G)$ 表示 G 的最小度。对于点的集合 $A,B\subseteq V(G)$,集合 A 与 B 之间的边集用 $E(A,B)$ 表示,边集 $E(A,B)$ 的数目用 $e(A,B)$ 表示。

设 $A(G)$ 是 G 的邻接矩阵,如果 $v_iv_j\in E(G)$,则矩阵 $A(G)$ 中的第 i 行第 j 列分量为1;反之,如果 $v_iv_j\notin E(G)$,则矩阵 $A(G)$ 中的第 i 行第 j 列分量为0. 设 $D(G)$ 是对角矩阵,其中,$D(G)$ 中的第 i 行第 j 列表示点 v_i 在图 G 的度。定义 G 的无符号拉普拉斯矩阵为 $Q(G)=D(G)+A(G)$,则 $Q(G)$ 的最大特征值 $\lambda_1(G)$ 为 $Q(G)$ 的无符号拉普拉斯谱半径,也是 G 的无符号拉普拉斯谱半径。

令 $x=(x_1,\cdots,x_n)^T\in\mathbb{R}^n$,$x_i$ 表示在点 v_i 处的取值。计算后得出

$$x^TQ(G)x=\sum_{v_iv_j\in E(G)}(x_i+x_j)^2$$

当 $x\neq0$ 并且对于任意点

$$v\in V(G)\text{有}(\lambda(G)-d(v))x(v)=\sum_{u\in N(v)}x(u)+d(v)$$

文献[1]~[6]给出了有关图的拉普拉斯谱半径界的相关结论,但是对于图的无符号拉普拉斯谱半径界的结论相对较少。目前,有关图的无符号拉普拉斯谱半径界的研究结果如下:

Cvetkovi'c 等人[7]:假设 G 是一个 n 阶连通图,并且 G 中各个点的度分别是 $d_1,d_2,\cdots,d_n$,则

$$2\min\{d_i|i=1,2,\cdots,n\}\leqslant\lambda_1(G)\leqslant2\max\{d_i|i=1,2,\cdots,n\}$$

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当且仅当G是正则图或半正则图时,等式成立。在证明过程中Cvetkovi′c等人[7]还研究得出以下结论:

假设G是一个n阶连通图,有:(1)$\lambda_1=0$当且仅当$E(G)=0$;(2)$0<\lambda_1<4$当且仅当G中的每个块都是路;(3)假设G是一个n阶连通图,则$\lambda_1=4$当且仅当$G\cong C_n$或$G\cong K_{1,3}$.

Cvetkovi′c等人[8]研究论证:假设G是一个n阶非空连通图时,则$\lambda_1(G)\geqslant\Delta(G)+1$当且仅当$G\cong K_{1,3}$时,等式成立。

Wang等人[9]研究论证:假设G是一个大小为m的n阶连通图,则

$$\lambda_1(G)\leqslant\frac{\delta(G)-1}{2}+\sqrt{2\left(\Delta^2(G)+\delta(G)\right)+2\left(2m-n\delta(G)\right)+\frac{\left(\delta(G)-1\right)^2}{4}}$$

当且仅当G是δ正则图时,等式成立。

Wang等人[9]研究论证:假设G是一个$n\geqslant 3$阶连通图,则

$$\lambda_1(G)\geqslant\frac{\Delta(G)+\delta(G)+\sqrt{\left(\Delta(G)-\delta(G)\right)^2+4\Delta(G)}}{2}$$

当且仅当$G\cong K_{1,n-1}$时,等式成立。

1 相关引理

引理1[8] 假设G是一个n阶连通图,并且G中各个点的度分别是$d_1,d_2,\cdots,d_n$,则

$$\min\{d_i+d_j|ij\in E(G)\}\leqslant\lambda_1(G)\leqslant\max\{d_i+d_j|ij\in E(G)\}$$

当G是正则图或半正则图时上述不等式取等。

引理2[11] 假设G是一个n阶连通图,H是G中的一个$m\leqslant n$阶子图,则对于$i=1,2,\cdots,m$,有$\lambda_i(H)\leqslant\lambda_i(G)$.

若无环图的顶点集能划分为两个非空子集X和Y,使得X中的任何两顶点之间无边相连并且Y中的任何两顶点之间无边相连,则称该图为二部图。

引理3 设H是各个点的度数为$c_1,\cdots,c_n$且边数$e(H)\geqslant 1$的二部图,则

$$\lambda_1(H)\geqslant\frac{1}{e(H)}\sum_{v_i\in V(H)}c_i^2$$

证明 令$x=(x_1,\cdots,x_n)^T$为一向量,其中$x_i=c_i(i=1,\cdots,n)$. 因为$e(H)\geqslant 1$,所以x是非零向量。根据*Rayleigh′s*定理,有

$$\lambda_1(H)\geqslant\frac{x^TQ(H)x}{x^Tx}=\frac{\sum\limits_{v_iv_j\in E(H)}\left(c_i+c_j\right)^2}{\sum\limits_{v_i\in V(H)}{c_i}^2}$$

再根据*Cauchy − Schwarz*不等式,则

$$\left(\sum_{v_i\in V(H)}{c_i}^2\right)^2=\left(\sum_{v_iv_j\in E(H)}1\cdot(c_i+c_j)\right)^2\leqslant e(H)\sum_{v_iv_j\in E(H)}\left(c_i+c_j\right)^2$$

所以,$\lambda_1(H)\geqslant\frac{1}{e(H)}\sum\limits_{v_i\in V(H)}c_i^2$.

2 连通图的无符号拉普拉斯谱半径的下界

定理1 如果G是最大度$\Delta(G)<n-1$的连通图,则

$$\lambda_1(G)\geqslant\max\left\{{m_i}'+\left(1+\frac{\left({m_i}'-1\right)^2}{d_{2,i}}\right)\frac{d_i}{{m_i}'}:v_i\in V(G)\right\}$$

其中,${m_i}'=\frac{\sum\limits_{v_iv_j\in E(G)}\left(d_j-|N_i\cap N_j|\right)}{d_i}$.

证明 令$v_i \in V(G)$并且H是点集为$V(G)$的G的子图，其中，子图H的边集$E(H)=E(N_i,V\setminus N_i)$. 设$c_i$表示在子图$H$中点$v_i$的度，因此$\sum_{v_j \in N_i} c_j = d_i m_i'$.

根据$Cauchy-Schwarz$不等式，有

$$\sum_{v_j \in N_i} c_j^2 \geqslant \frac{1}{\sum_{v_j \in N_i} 1}\left(\sum_{v_j \in N_i} c_j\right)^2 = d_i (m_i')^2$$

设$N_{2,i}$表示G中与点v_i距离为2的点的集合。因为G是连通的并且$d_i < n-1$，所以$d_{2,i} \geqslant 1$. 又因为$\sum_{v_j \in N_{2,i}} c_j = d_i m_i' - d_i$，因此得到

$$\sum_{v_j \in V\setminus N_i} c_j^2 = d_i^2 + \sum_{v_j \in N_{2,i}} c_j^2 \geqslant d_i^2 + d_{2,i}\left(\frac{d_i m_i' - d_i}{d_{2,i}}\right)^2 = \left(1 + \frac{(m_i'-1)^2}{d_{2,i}}\right) d_i^2$$

再根据引理2，有

$$\lambda_1(G) \geqslant \lambda_1(H) \geqslant \frac{1}{e(N_i, V\setminus N_i)} \sum_{v_j \in V(H)} c_j^2 \geqslant \frac{1}{d_i m_i'}\left(d_i m_i'^2 + \left(1 + \frac{(m_i'-1)^2}{d_{2,i}}\right) d_i^2\right) = m_i' + \left(1 + \frac{(m_i'-1)^2}{d_{2,i}}\right)\frac{d_i}{m_i'}$$

最后有$\lambda_1(G) \geqslant \max\left\{ m_i' + \left(1 + \frac{(m_i'-1)^2}{d_{2,i}}\right)\frac{d_i}{m_i'} : v_i \in V(G)\right\}$.

定理2 如果G是连通图，则

$$\lambda_1(G) \geqslant \max\left\{\left(p_{ij} + \frac{(1-p_{ij})^2}{p_{ij}\max\{1, d_i - 1\}}\right)|N_i \cup N_j| : v_i v_j \in E(G), d_i \geqslant d_j\right\}$$

其中，$p_{ij} = \frac{e(N_i, N_j - N_i)}{d_i\left(|N_i \cup N_j| - d_i\right)}$.

证明 设$v_i v_j \in E(G)$并且$d_i \geqslant d_j$. 令H是G的子图，其中子图H的边集$E(H)=E(N_i, N_j \setminus N_i)$. 对于任意的点$v_k \in V(H)$，用$c_k$表示点$v_k$在图$H$中的度。令$e(N_i, N_j \setminus N_i) = t$，$|N_i \cup N_j| = N$，则$p_{ij} = \frac{t}{d_i(N-d_i)}$. 为了表达简洁，用$p$表示$p_{ij}$. 如果$N - d_i = 1$，则$p=1$，这说明此时$G$有一最大度为$n-1$的子图$G^*$，且$G^*$是星图。根据引理1可以得出$\lambda(G) \geqslant d_i + 1$. 接下来假设$N - d_i \geqslant 2$，于是就有$d_i \geqslant 2$.

根据引理3，有

$$\lambda_1(H) \geqslant \frac{1}{e(H)} \sum_{v_i \in V(H)} c_i^2 = \frac{1}{t}\sum_{v_k \in N_i} c_k^2 + \frac{1}{t}\sum_{v_k \in N_j \setminus N_i} c_k^2$$

根据$Cauchy-Schwarz$不等式，有

$$\begin{aligned}
\frac{1}{t}\sum_{v_k \in N_i} c_k^2 &\geqslant \frac{1}{t}\left(c_j^2 + (d_i - 1)\left(\frac{t-(N-d_i)}{d_i - 1}\right)^2\right) \\
&= \frac{1}{t}\left((N-d_i)^2 + \frac{(t-(N-d_i))^2}{d_i - 1}\right) \\
&= \frac{N-d_i}{pd_i} + \frac{t}{d_i - 1} + \frac{(N-d_i)}{pd_i(d_i-1)} - \frac{2(N-d_i)}{d_i - 1} \\
&= (N-d_i)\left(\frac{1}{pd_i} + \frac{pd_i}{d_i - 1} + \frac{1}{pd_i(d_i-1)} - \frac{2}{d_i - 1}\right)
\end{aligned}$$

$$= (N-d_i)\left(p+\frac{1}{p(d_i-1)}+\frac{p-2}{d_i-1}\right)$$
$$= (N-d_i)\left(p+\frac{(1-p)^2}{p(d_i-1)}\right)$$

并且

$$\frac{1}{t}\sum_{v_k\in N_j\setminus N_i}c_k^2 \geqslant \frac{1}{t}\left(c_i^2+(N-d_i-1)\left(\frac{t-d_i}{N-d_i-1}\right)^2\right)$$
$$=\frac{1}{t}\left(d_i^2+\frac{(t-d_i)^2}{N-d_i-1}\right)$$
$$=\frac{d_i}{p(N-d_i)}+\frac{t}{N-d_i-1}+\frac{d_i}{p(N-d_i)(N-d_i-1)}-\frac{2d_i}{N-d_i-1}$$
$$=\left(\frac{1}{p(N-d_i)}+\frac{p(N-d_i)}{N-d_i-1}+\frac{1}{p(N-d_i)(N-d_i-1)}-\frac{2}{N-d_i-1}\right)d_i$$
$$=\left(p+\frac{1}{p(N-d_i-1)}+\frac{p-2}{N-d_i-1}\right)d_i$$
$$=\left(p+\frac{(1-p)^2}{p(N-d_i-1)}\right)d_i$$

根据引理2,再利用$d_i \geqslant d_j$且$N = d_i + d_j - |N_i \cap N_j|$有

$$\lambda_1(G) \geqslant \lambda_1(H)$$
$$\geqslant \frac{1}{t}\left(\sum_{v_k\in N_i}c_k^2+\sum_{v_k\in N_j\setminus N_i}c_k^2\right)$$
$$\geqslant (N-d_i)\left(p+\frac{(1-p)^2}{p(d_i-1)}\right)+\left(p+\frac{(1-p)^2}{p(N-d_i-1)}\right)d_i$$
$$\geqslant (N-d_i)\left(p+\frac{(1-p)^2}{p(d_i-1)}\right)+\left(p+\frac{(1-p)^2}{p(d_i-1)}\right)d_i$$
$$=\left(p+\frac{(1-p)^2}{p(d_i-1)}\right)N$$

综上所述,最终得到

$$\lambda_1(G) \geqslant \max\left\{\left(p_{ij}+\frac{(1-p_{ij})^2}{p_{ij}\max\{1,d_i-1\}}\right)|N_i \cup N_j|:v_iv_j \in E(G),d_i \geqslant d_j\right\}$$

3 连通图的无符号拉普拉斯谱半径的上界

定理3 假设G是n连通图,则

$$\lambda_1(G) \leqslant \sqrt{2\Delta(G)(\Delta(G)-1)+1}+1$$

证明 令$x = (x_1,\cdots,x_n)^T$为无符号拉普拉斯谱半径$\lambda_1(G)$对应的特征向量,可以假设向量x中某个分量$x_i = 1$,其余的分量$|x_k| \leqslant 1(k = 1,\cdots,i-1,i+1,\cdots,n)$. 令$c_{ij}$表示点$v_i$和点$v_j$的公共邻点数目,即$c_{ij} = |N_i \cap N_j|$.

由假设知$|x_k| \leqslant 1(k = 1,2,\cdots,n)$,因此

$$\sum_{v_iv_k\in E(G),v_jv_k\notin E(G)} x_k \leqslant d_i - c_{ij}$$

$$\sum_{v_jv_k\in E(G),v_iv_k\notin E(G)} x_k \leqslant d_j - c_{ij}$$

$$\sum_{v_iv_k\in E(G),v_jv_k\in E(G)} x_k \leqslant c_{ij}$$

根据 $Qx = \lambda_1 x$,有

$$\lambda_1(G)x_i = d_ix_i + \sum_{v_kv_i\in E(G)} x_k = d_ix_i + \sum_{v_kv_i\in E(G),v_kv_j\in E(G)} x_k + \sum_{v_kv_i\in E(G),v_kv_j\notin E(G)} x_k$$

又因为 $x_i = 1$,得

$$\lambda_1(G) = d_i + \sum_{v_kv_i\in E(G),v_kv_j\in E(G)} x_k + \sum_{v_kv_i\in E(G),v_kv_j\notin E(G)} x_k$$

同理可得

$$\lambda_1(G)x_j = d_jx_j + \sum_{v_kv_i\in E(G),v_kv_j\in E(G)} x_k + \sum_{v_kv_i\notin E(G),v_kv_j\in E(G)} x_k$$

综上,可得

$$\begin{aligned}
\lambda_1(G)(1 + x_j) &= \lambda_1(G) + \lambda_1(G)x_j \\
&= d_i + \sum_{v_kv_i\in E(G),v_kv_j\in E(G)} x_k + \sum_{v_kv_i\in E(G),v_kv_j\notin E(G)} x_k + d_jx_j + \sum_{v_kv_i\in E(G),v_kv_j\in E(G)} x_k + \sum_{v_kv_i\notin E(G),v_kv_j\in E(G)} x_k \\
&\leqslant d_i + c_{ij} + (d_i - c_{ij}) + d_j + c_{ij} + (d_j - c_{ij}) \\
&= 2(d_i + d_j)
\end{aligned}$$

将按上述式子,关于 j 且满足 $v_iv_j \in E(G)$ 于不等式两侧进行求和,得

$$\lambda_1(G)\left(d_i + \sum_{v_iv_j\in E(G)} x_j\right) \leqslant 2\left(d_i^2 + \sum_{v_iv_j\in E(G)} x_j\right)$$

又因为

$$\lambda_1(G) = d_i + \sum_{v_iv_j\in E(G)} x_j$$

所以

$$\lambda_1^2(G) \leqslant 2(d_i(d_i - 1) + \lambda_1(G))$$

变形可得

$$(\lambda_1(G) - 1)^2 \leqslant 2d_i(d_i - 1) + 1$$

因此

$$\lambda_1(G) \leqslant \max_{v_i\in V(G)}\left\{\sqrt{2d_i(d_i - 1) + 1}\right\} + 1 = \sqrt{2\Delta(G)(\Delta(G) - 1) + 1} + 1$$

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The Bound for the Signless Laplacian Spectral Radius of Connected Graph

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Abstract: Let $G=(V,E)$ be the simple connected graph of order n and $\lambda_1(G)$ be the spectral radius of signless Laplacian of G. For each $v_i \in V(G)$, the set of its neighbors in G is denoted by N_i, and the number of neighbors of v_i in G is denoted by d_i. The maximum degree of G is defined by $\Delta(G)$. In this paper, it is showm that if G is connected graph and $\Delta(G) < n-1$, then $\lambda_1(G) \geqslant \max\left\{ m_i' + \left(1 + \frac{(m_i'-1)^2}{d_{2,i}}\right)\frac{d_i}{m_i'} : v_i \in V(G) \right\}$, where $m_i' = \frac{\sum_{v_i v_j \in E(G)} \left(d_j - |N_i \cap N_j|\right)}{d_i}$ and $d_{2,i}$ is the number of vertices at distance two from v_i in G. Also it is shown that $\lambda_1(G) \geqslant \max\left\{ \left(p_{ij} + \frac{(1-p_{ij})^2}{p_{ij}\max\{1, d_i - 1\}}\right) |N_i \cup N_j| : v_i v_j \in E(G), d_i \geqslant d_j \right\}$, Where $p_{ij} = \frac{e(N_i, N_j - N_i)}{d_i\left(|N_i \cup N_j| - d_i\right)}$. Then it is also shown that if G is a connected graph, then $\lambda_1(G) \leqslant \sqrt{2\Delta(G)(\Delta(G)-1)+1} + 1$.

Keywords: Signless Laplacian Matrix; Spectral radius; Connected graph